J. Fluid Mech. (1973), vol. 60, part 4, pp. 801–811
Printed in Great Britain

Mean motions and impulse of a guided internal gravity wave packet

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(Received 12 January 1972 and in revised form 22 March 1973)

Second-order mean fields of motion and density are calculated for the two-dimensional problem of an internal gravity wave packet (the waves are predominantly of a single frequency ω and wavenumber k) propagating as a wave-guide mode in an inviscid, diffusionless Boussinesq fluid of constant buoyancy frequency N, confined between horizontal boundaries. (The same mathematical analysis applies to the formally identical problem for inertia waves in a homogeneous rotating fluid.)

To leading order the mean motions turn out to be zero outside the wave packet, which consequently possesses a well-defined fluid impulse $\mathscr{I}$. This is directed horizontally, and is given in magnitude and sense by

$$\mathscr{I} = \alpha\mathscr{M}; \quad \alpha = \frac{2c_g(c-c_g)(c+2c_g)}{c^3-4c_g^3}.$$

Here $\mathscr{M}$ is the so-called 'wave momentum', defined as wave energy divided by horizontal phase velocity $c \equiv \omega/k$, and $c_g = c(N^2-\omega^2)/N^2$, the group velocity.

If the wave packet is supposed generated by a horizontally towed obstacle, $\mathscr{M}$ appears as the total fluid impulse, but of this a portion $\mathscr{M}-\mathscr{I}$ in general propagates independently away from the wave packet in the form of long waves. When the wave packet itself is totally reflected by a vertical barrier immersed in the fluid, the time-integrated horizontal force on the barrier equals $2\mathscr{I}$ (and not $2\mathscr{M}$ as might have been expected from a naive analogy with the radiation pressure of electromagnetic waves.)

1. Introduction

The suggestion that waves in fluids should 'possess' a well-defined amount of momentum, like photons, can be traced back to Poynting (1905) and earlier. That this analogy with photons, taken literally, cannot be generally applicable to waves in material media – in contrast to the usually valid idea of a radiation stress whereby waves bring about a *transfer* of momentum – has been clearly pointed out by Brillouin (1925, 1964). Nevertheless the photon analogy tends to persist in the fluid dynamical literature. A reason seems to be that it can lead to correct or partly correct conclusions in certain problems, such as those considered by Bretherton (1969), Holton (1970), Lindzen (1971), and Matsuno (1971);

but see Hasselmann (1971) for discussion of an oceanographically important problem (concerning deep-water surface gravity waves) in which a recent application of the photon analogy led to a completely incorrect conclusion.

The purpose of this note is to record an example which (*a*) complements those considered by Bretherton (1969), and (*b*) further illustrates the way in which the photon analogy can sometimes appear valid and at the same time mislead. The results are not new in principle, since similar results hold for surface gravity waves in water of finite depth and can be found from the analysis of Longuet-Higgins & Stewart (1962). But the present example is mathematically and physically simpler.

Unlike surface gravity waves, the internal waves in the present example always possess *zero* momentum, at the level of approximation considered, but a wave packet of wavenumber k turns out to have a definite fluid impulse $\mathscr{I}$. The way in which the value of $\mathscr{I}$ depends on k, however, is very different from what might be expected from an analogy to photons. Such an analogy might, for example, lead one to expect that an immersed obstacle reflecting a wave packet incident from the left would feel a force toward the right; but in fact the force can equally well be toward the left (§3.2). Furthermore a wave *train*, involving a significant spread of wavenumbers, does not appear to have a well-defined impulse at all, let alone that impulse which would be obtained by regarding the wave train as a superposition of wave packets and summing the corresponding values of $\mathscr{I}$.†

2. Analysis

2.1. *Equations, momentum, and impulse*

Consider two-dimensional incompressible flow, in which all fields depend upon the Cartesian co-ordinates x and z only, and satisfy

$$\mathbf{u}_t + \mathbf{u}.\nabla\mathbf{u} = -\nabla(p/\rho) + \hat{\mathbf{z}}\theta, \tag{1a}$$

$$\nabla.\mathbf{u} = 0, \tag{1b}$$

$$\theta_t + \mathbf{u}.\nabla\theta + N^2 w = 0 \tag{1c}$$

$$[\mathbf{u} \equiv (u, 0, w);\ \nabla \equiv (\partial/\partial x, 0, \partial/\partial z)].$$

$\hat{\mathbf{z}}$ is a unit vector in the z-direction. These equations represent either a Boussinesq, stably stratified fluid under gravity $(0, 0, -g)$, with buoyancy frequency N, here assumed constant, or a homogeneous fluid in a frame of reference rotating with constant angular velocity $(\frac{1}{2}N, 0, 0)$. In the 'stratified' interpretation, which we use for descriptive purposes, $\mathbf{u}$ is the velocity and θ the buoyancy acceleration. The latter is defined as $-g$ times the fractional departure of the local density from the basic density $\rho(1-g^{-1}N^2 z)$, where ρ is the (constant) mean density. In the 'rotating' interpretation the velocity field is $\mathbf{u} + (0, -N^{-1}\theta, 0)$ in our notation. The presence of the horizontal boundaries is expressed by

$$w = 0 \quad \text{on} \quad z = 0,\ H. \tag{1d}$$

† For brevity the mathematical argument leading to this latter, negative conclusion will not be reproduced below. But the fact that the circumstances in which $\mathscr{I}$ exists are special will appear plausible from the analysis given below in §2.2.

A consequence of (1*b*) and (1*d*) is that the total horizontal momentum per unit y-distance is zero:

$$\rho\int_{-\infty}^{\infty}\int_{0}^{H} u\,dz\,dx = 0, \tag{2}$$

for any disturbance of finite extent in x. (In the case of stratified fluid there is a further contribution to the momentum due to the effect of the density variations $\Delta\rho$ on the fluid inertia per unit volume. But, in comparison with the impulse to be defined in (3), this effect becomes arbitrarily small in the Boussinesq limit, $\Delta\rho/\rho \to 0$ with $g\Delta\rho/\rho$ finite. Alternatively, one could avoid discussion of the Boussinesq approximation by restricting attention to the 'rotating' interpretation.)

The two-dimensional form of the horizontal component of fluid *impulse* can be defined, as

$$\mathscr{P} \equiv \rho\int_{-\infty}^{\infty}\int_{0}^{H} -z\eta\,dz\,dx \quad (\eta \equiv u_z - w_x),$$

$$= \rho\iint -zu_z\,dz\,dx = -\rho\int_{-\infty}^{\infty}[zu]_0^H\,dx, \tag{3}$$

provided that

$$\iint \eta\,dz\,dx = \int_{-\infty}^{\infty}[u]_0^H\,dx = 0 \tag{4}$$

(Lamb 1932, §157; Batchelor 1967, equation (7.3.7) etc.; Benjamin 1970, §3). $\mathscr{P}$ is a constant of the motion, under (1), and can be shown to be equal to minus the resultant horizontal component of a set of impulsive body forces which would instantaneously annul the motion, provided that these forces can be supposed to be sufficiently localized. For instance, it is sufficient that they act within a finite region of the x, z plane and that they, or their x-averaged horizontal component, tend to zero as the boundaries $z = 0, H$ are approached. In that case the motion necessarily satisfies (4).

2.2. *Calculation of $\mathscr{P}$ for an isolated wave packet*

Let a be a small dimensionless parameter characterizing the amplitude of the waves, and μ a small dimensionless parameter characterizing slowness of variation of the complex amplitude relative to the space and time scales H and N^{-1}. Defining the slow variables

$$X = \mu x, \quad T = \mu t,$$

consider the wave packet described by

$$u = a\ \mathrm{Re}\ [U(X,T)\,e^{ik(x-ct)}]\cos(mz), \tag{5a}$$

$$w = -am^{-1}\ \mathrm{Re}\ [(ikU+\mu U_X)\,e^{ik(x-ct)}]\sin(mz)\{1+O(\mu^2)\}, \tag{5b}$$

$$p = a\rho c\,\mathrm{Re}\,[(U+i\mu k\kappa^{-2}U_X)\,e^{ik(x-ct)}]\cos(mz)\{1+O(\mu^2)\}, \tag{5c}$$

$$\theta = -aN^2(mc)^{-1}\,\mathrm{Re}\,[(U-i\mu k\kappa^{-2}U_X)\,e^{ik(x-ct)}]\sin(mz)\{1+O(\mu^2)\}. \tag{5d}$$

Here

$$\kappa \equiv (k^2+m^2)^{\frac{1}{2}} > 0,$$

and suffixes denote partial differentiation. To satisfy the boundary conditions (1d), the vertical wavenumber

$$m = n\pi/H,$$

where n is a positive integer. The horizontal wavenumber k is taken to be constant. It can easily be verified that (5) satisfies the linearized form of equations (1), correct to the first two orders in μ, provided first that the dispersion relation is satisfied, i.e. that

$$c = \pm N/\kappa, \tag{6}$$

and second that the envelope $U(X, T)$ satisfies

$$U_T = -c_g U_X\{1+O(\mu)\}, \tag{7}$$

where the group velocity

$$c_g = \partial(ck)/\partial k = cm^2/\kappa^2.$$

U may be complex-valued, and together with a sufficient number of its derivatives U_X, U_T, U_{XX}, etc., will be assumed to tend to zero as $|X|$ becomes large compared to H, i.e. as $|x|$ becomes large compared to $\mu^{-1}H$.†

Now if (5a) is substituted into (3) the oscillations tend to average out, the result for instance being $O(\mu^{M-1})$ if it is assumed that

$$\int_{-\infty}^{\infty}\left|\frac{\partial^M U}{\partial X^M}\right| dX < \infty.$$

Thus to obtain a contribution to $\mathscr{P}$ that does not vanish with μ, we must go to second order in the amplitude a. We shall be interested only in the 'mean', or slowly varying, part $\bar{\mathbf{u}}(X, z, T)$, $\bar{p}(X, z, T)$, $\bar{\theta}(X, z, T)$ of the order-a^2 solution, where $\overline{(\;)}$ denotes the Eulerian averaging operation over a wave period $2\pi(ck)^{-1}$. This operation, by definition, is carried out at a fixed point in space and obliterates all terms containing oscillatory factors such as $e^{\pm 2ik(x-ct)}$.

The *Reynolds stress* associated with the Eulerian averaging operation $\overline{(\;)}$ is first calculated from (5a) and (5b):

$$\left.\begin{aligned} -\rho\overline{u^2} &= -2c_g(E/c)\cos^2(mz),\\ -\rho\overline{uw} &= \mu c_g m^{-1}(E_X/c)\cos(mz)\sin(mz)\{1+O(\mu)\},\\ -\rho\overline{w^2} &= -2c_g k^2 m^{-2}(E/c)\sin^2(mz)\{1+O(\mu)\},\end{aligned}\right\} \tag{8}$$

where

$$E \equiv \tfrac{1}{4}\rho a^2|U|^2 c/c_g, \tag{9}$$

which is the leading contribution to the wave energy density defined as

$$1/H\int_0^H \tfrac{1}{2}\rho(\overline{u^2}+\overline{w^2}+N^{-2}\overline{\theta^2})\,dz.$$

† It should be noted that variable amplitude is an essential ingredient in our problem. If U were a constant then (5a, b, d) would represent an exact solution of (1) for finite a (Long 1955), since then $\mathbf{u}.\nabla\theta = 0$ and $\mathbf{u}.\nabla\mathbf{u}$ is irrotational. For this solution, $\mathscr{P}$, as well as the Boussinesq momentum (2), is exactly zero when averaged over a wavelength (Benjamin 1970, equation (4.17)) – in contrast to analogous problems in surface gravity waves.

Taking the divergence of the Reynolds stress to obtain $-\rho\overline{\mathbf{u}.\nabla\mathbf{u}}$, and averaging (1 *a*), we obtain the Eulerian-mean momentum equations, to leading order in a and μ. The horizontal component yields

$$\rho\bar{u}_T+\bar{p}_X=-c_g E_X/c, \tag{10a}$$

and the vertical component

$$-\rho\bar{\theta}+\bar{p}_z=-2c_g k^2 m^{-1}(E/c)\sin(2mz). \tag{10b}$$

The mean continuity equation can be written

$$\mu\bar{u}_X+\bar{w}_z=0, \tag{10c}$$

implying that $\bar{w}=\bar{u}\times O(\mu)$, whence neglect of the vertical acceleration $\mu\bar{w}_T$ in (10*b*). The *buoyancy flux* $\rho\overline{\mathbf{u}\theta}$ is obtained from (5*a*, *b*, *d*) as

$$c_g[-\kappa^2 m^{-1}E\sin(2mz),\ 0,\ \mu E_X\sin^2(mz)]\{1+O(\mu)\}.$$

Taking minus the divergence of this to obtain $-\rho\overline{\mathbf{u}.\nabla\theta}$, we find from (1*c*) that, to leading order in a and μ,

$$\rho(\bar{\theta}_T+\mu^{-1}N^2\bar{w})=c_g k^2 m^{-1}E_X\sin(2mz). \tag{10d}$$

In virtue of the steadily translating character of the forcing terms on the right – this appears to be the most essential way in which the constancy of c, k and c_g is used in the analysis – equations (10) have the following particular integral. It satisfies the boundary condition $\bar{w}=0$ at $z=0$, H and, to leading order, vanishes outside the wave packet:

$$\rho\bar{u}=-\alpha(E/c)\cos(2mz), \tag{11a}$$

$$\rho\bar{w}=\mu\alpha(2m)^{-1}(E_X/c)\sin(2mz), \tag{11b}$$

$$\bar{p}=-\alpha c_g(E/c)\cos(2mz)-c_g E/c, \tag{11c}$$

$$\rho\bar{\theta}=\beta m E\sin(2mz). \tag{11d}$$

We have used the fact that, from (7) and (9), $E(X,T)$ satisfies

$$E_T=-c_g E_X\{1+O(\mu)\}; \tag{12}$$

α and β are dimensionless quantities that depend upon c_g/c only, in the manner shown graphically in figure 1, and defined by

$$\alpha\equiv\frac{2c_g(c-c_g)}{c^3-4c_g^3}\{2c_g+c\}, \tag{13a}$$

$$\beta\equiv\frac{2(c-c_g)}{c^3-4c_g^3}\{c^2+2c_g^2\}. \tag{13b}$$

In each of these expressions the second term within curly brackets can be traced back to the forcing term on the right of (10*d*), minus divergence of buoyancy flux. The other, first, term within the curly brackets is entirely due to the contribution $-(\overline{w^2})_z$ to the vertical force (the other contribution $-\mu(\overline{uw})_X$ being negligible). The horizontal force, μ times the right-hand side of (10*a*), is independent of z and is balanced entirely by the gradient of the second contribution to

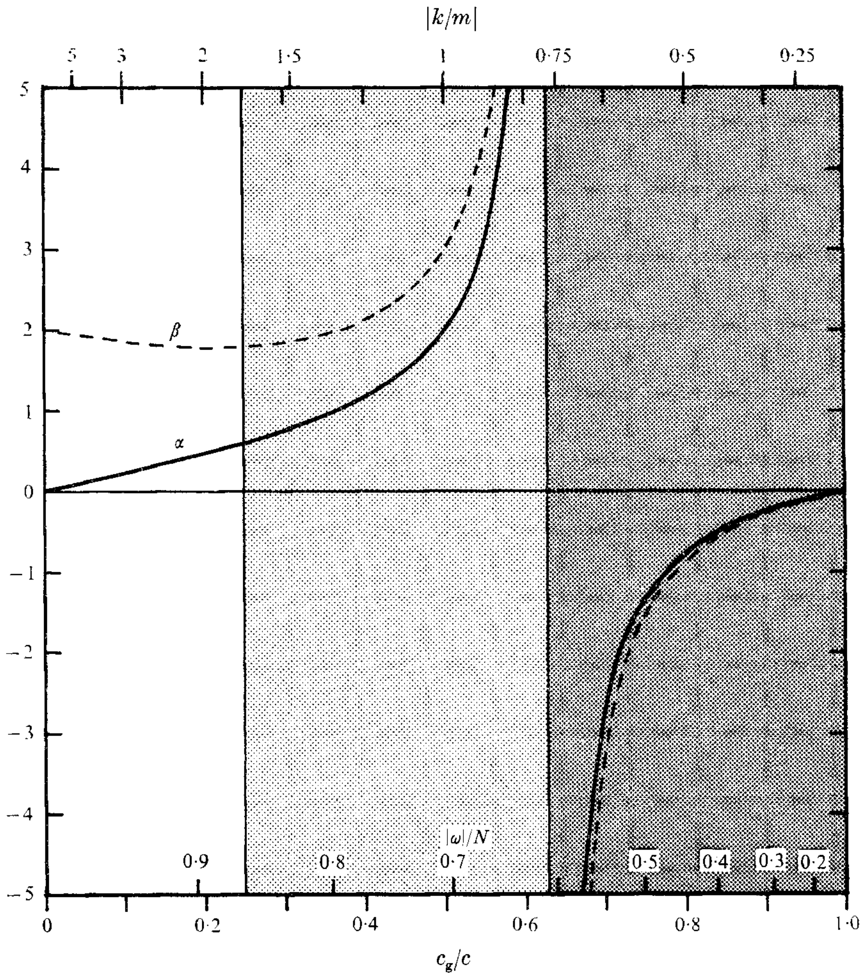

FIGURE 1. Dependence of $\alpha\,(=\mathscr{I}/\mathscr{M}$: solid curve) and β (broken curve) upon c_g/c, or upon wave aspect ratio $|k/m|$ or dimensionless frequency $|\omega|/N$. The shading is intended to show the correspondence with figure 4 of McIntyre (1972) (in which the abscissa is $(c_g/c)^{\frac{1}{2}}$, in the present notation).

$\overline{p}$ in (11*c*). The relatively trivial role played by the horizontal force contrasts with the situation in Bretherton's examples (1969, q.v.); the difference is due to our having postulated two-dimensional motion and horizontal boundaries.

The corresponding Lagrangian-mean motion is not directly relevant to the calculation of $\mathscr{I}$; it is given in the appendix, together with some incidental remarks concerning the relevance to this problem of the 'radiation stress' concept.

'Free' solutions $\overline{u} = \text{func}\,(x \pm C_{n'}t)\cos(m'z)$, etc., i.e. solutions of the homogeneous counterparts of equations (10), may be added to (11). But such contributions propagate with the long-wave speeds

$$C_{n'} \equiv N/m' \quad (\neq c_g \text{ in general}) \quad [m' = n'\pi/H;\; n' = 1, 2, \ldots] \tag{14}$$

and so will not in general remain associated with the wave packet.† Unlike the particular integral (11), such additional contributions depend upon the initial conditions, as will be illustrated shortly.

Substituting (11a) into (3), and noting that (4) is satisfied, we obtain the impulse of the wave packet correct to order $a^2\mu$:

$$\begin{aligned}\mathscr{P} &= -\rho\int_{-\infty}^{\infty}[z\bar{u}]_0^H\,dx \\ &= \alpha\mathscr{M} \equiv \mathscr{I},\quad \text{say,}\end{aligned} \tag{15}$$

where

$$\mathscr{M} \equiv \mathscr{E}/c \quad \text{and} \quad \mathscr{E} \equiv H\int_{-\infty}^{\infty}E\,dx,$$

the total wave energy.

3. Discussion

3.1. *Generation of the waves by an obstacle towed at speed c*

If the wave packet is supposed generated by steadily towing an obstacle through the fluid, the time integral of the horizontal wave drag on the obstacle is $\mathscr{M}$ rather than $\mathscr{I}$, as can be seen from energy considerations. It can be shown (Benjamin 1970, §3) that the total impulse is then $\mathscr{M}$, and it is of interest to see how only $\mathscr{I}$ of this comes to be associated with the wave packet after generation (in contrast to what might tacitly be assumed on the basis of the photon analogy).

A detailed analysis of this towing problem has been given elsewhere (McIntyre 1972, referred to below as M), and solutions to corresponding problems for surface and interfacial gravity waves have been given by Benjamin (1970, §2) and Keady (1971). Here we confine ourselves to a brief summary of those results which are relevant to the present discussion.

Figure 2(a) schematically depicts a wave packet being generated by a two-dimensional body which was introduced into the fluid at point A at $t = 0$, moving with constant speed c. In the analysis it was assumed that the body is slender, and slides along one boundary. For simplicity we suppose that $C_2 < c < C_1$ (see (14)), so that only waves in the lowest mode $n = 1$ are continually generated. Their presence is indicated in figure 2 by the disturbance to the central 'dye streak'.

At second order in a, there is also present a pair of 'columnar disturbances' with the modal structure $n = 2$. The fine streamlines represent schematically the associated contribution to the Eulerian-mean velocity field. The corresponding buoyancy field $\bar{\theta}$ is not depicted. The nonlinear terms forcing these disturbances

† An exception is the singular case $c_g/c = 4^{-\frac{1}{3}} = 0{\cdot}630$, corresponding to violation of (14) for $m' = 2m$. The wave aspect-ratio $|k/m|$ is then $0{\cdot}766$ and the dimensionless frequency $|\omega|/N = 0{\cdot}608$. Reconsideration of (10) shows that then, as would be expected, the second-order mean motion grows resonantly as t. [This effect is not described by the usual resonant-interaction theory (e.g. Martin *et al.* 1972), since it disappears if U_X is formally set to zero.]

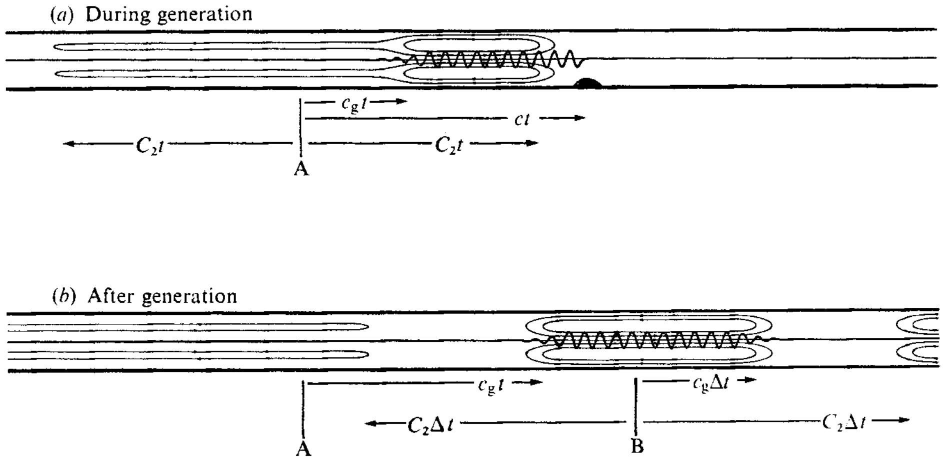

FIGURE 2. (*a*) Sketch (see text) showing generation of columnar disturbances during wave generation. The horizontal scale is compressed and the wave amplitude exaggerated. (*b*) Situation after wave generation, showing part of the columnar disturbances separating off as independently propagating, long-wave transients. Here $\Delta t = t - t_B$. (In the drawing, $0 < c_g < C_2 < c < C_1$. This implies $\frac{1}{4} < c/c_g < 4^{-\frac{1}{3}}$, corresponding to the lightly shaded part of figure 1 and positive $\alpha = \mathscr{I}/\mathscr{M}$.)

are significant mainly at the left-hand end of the developing wave packet (but not at the right-hand end during wave generation). The free ends of the two columnar disturbances propagate with the long-wave speed C_2, so that a continually lengthening region contains steady, horizontal, second-order velocities, given by the asymptotic formulae (4.19*a*) and (4.17*a*) of M. In the present dimensional notation these formulae are equivalent to

$$\rho\bar{u} \sim -(c-c_g)\frac{E}{c}\left\{\frac{2C_2-c}{4C_2|C_2+c_g|} \quad \text{and} \quad \frac{2C_2+c}{4C_2|C_2-c_g|}\right\}\cos\left(\frac{2\pi z}{H}\right) \tag{16}$$

for the left- and right-hand columnar disturbances respectively. Here the sign convention is different from that in M, in order to make the positive senses for $\bar{u}$ and c the same. Note that $(c-c_g)$ and the two expressions within curly brackets are positive. From these formulae and the fact that $\mathscr{E} \sim EH(c-c_g)\,t$ we see that, asymptotically for large Nt, (4) is satisfied and the total impulse is $\mathscr{E}/c = \mathscr{M}$, as expected.

When the towed body is removed, at point B and time t_B, say, the total impulse becomes constant and the right-hand end of the wave packet begins to generate a second pair of columnar disturbances, the same in magnitude but opposite in sign to the first pair. After a while, cancellation takes place outside the wave packet where corresponding disturbances overlap, and we are left with the situation shown in figure 2(*b*). The second-order mean motions have split into three parts, separated by distances that continually increase as t. The part that stays with the wave packet has impulse $\mathscr{I}$ (to an approximation valid for large Nt_B). The two freely propagating parts, which ultimately occupy regions remote from the wave packet, must therefore have impulse $\mathscr{M}-\mathscr{I}$ approximately; and this conclusion may be checked directly from (16), or from (M 4.20).

The amount of impulse $\mathscr{M}-\mathscr{I}$ contained in the second-order, freely propagating long waves depends on the particular way in which the waves were generated and cannot, like $\mathscr{I}$, be regarded as a property of the waves themselves. For instance, the waves could have been generated by a moving heat (buoyancy) source, in which case we should find long-wave transients containing impulse $-\mathscr{I}$, since a trivial extension of Benjamin's analysis (1970, §3) shows that the total impulse would then be zero.

We reiterate that the clear separation of the total impulse into a part associated with the waves plus an independent, transient part, seems to be a special result and does not generalize, for instance, to the case where k and c are allowed to be functions of X and T. It then appears that the second-order disturbances radiated from the various parts of the wave train need not cancel outside it.

3.2. *Force on a barrier reflecting the waves*

If a fixed, two-dimensional obstacle is immersed in the fluid, Benjamin's analysis (1970, §3) yields an impulse principle for the integral with respect to time, $\langle\mathscr{F}\rangle$, of the horizontal component of the force on the obstacle due to any unsteady disturbance. The principle states that $\langle\mathscr{F}\rangle$ equals the change in $\mathscr{P}$ over any period of time at the beginning and end of which the fluid at the obstacle is undisturbed.† Together with the present results this suggests the simple rule that reflexion, by the obstacle, of a fraction λ of the energy of an incident wave packet of impulse $\mathscr{I}$ will give

$$\langle\mathscr{F}\rangle = 2\lambda\mathscr{I}. \tag{17}$$

For obstacles of arbitrary shape this is only a conjecture, since we have not ruled out the (albeit somewhat remote) possibility that second-order long waves carrying significant impulse might be generated as a result of the reflexion process.

Equation (17), has, however, been verified by direct calculation for the simple limiting case of a vertical barrier which occupies (nearly) the whole height $0 < z < H$ of the channel and totally reflects the wave packet ($\lambda = 1$). The analysis shows that, in this case, no significant long-wave radiation occurs during reflexion. The solution, not reproduced here, consists of the incident wave packet and its image, described correct to order a^2, plus the second-order response to their mutual interaction. The latter is found to contribute

$$2c_g(E/c)\{1+O(\mu)\}$$

to the time-averaged pressure at the barrier. This just cancels the other z-independent contribution, namely twice the last term of (11*c*) – as might have been anticipated from the vanishing of the total momentum, (2). (The cancellation is a further reason for regarding the last term of (11*c*) as physically uninteresting in this problem.)

The only remaining contribution to the mean pressure at the barrier is twice

† In contrast with the case of a steadily towed obstacle, it seems generally necessary to exclude obstacles in contact with the boundary, apparently because (4) might then be violated in consequence of systematically differing values of θ at opposite sides of such an obstacle.

the first term of (11*c*). Using the condition that the barrier 'nearly' touches the boundaries, to remove the ambiguity in the pressure on the far side, we immediately verify (17) with $\lambda = 1$.

The writer is indebted to D. G. Andrews, F. P. Bretherton, W. L. Jones, O. M. Phillips and the referees for the paper for helpful comments on a first version. He also thanks St John's College, Cambridge, for support in the form of a research fellowship, and the Institute of Meteorology, University of Stockholm, for hospitality during the writing of this paper.

Appendix. The Lagrangian-mean flow and the 'radiation stress' concept

To first order in a the particle displacement $\boldsymbol{\xi}(x,z,t) \equiv \{\xi, 0, \zeta\}$ is defined by

$$\xi_t = u, \quad \zeta_t = w, \quad \bar{\xi} = \bar{\zeta} = 0.$$

The Stokes drifts, to leading order in μ, are found to be

$$\bar{u}^{\mathrm{S}} \equiv \overline{\boldsymbol{\xi}.\nabla u} = (2Ec_{\mathrm{g}}/\rho c^2)\cos(2mz) \tag{A 1a}$$

(Thorpe 1968, p. 612), and

$$\bar{w}^{\mathrm{S}} \equiv \overline{\boldsymbol{\xi}.\nabla w} = \frac{-\mu c_{\mathrm{g}}}{\rho c^3 m}(cE_X - E_T)\sin(2mz). \tag{A 1b}$$

It is noteworthy that the Stokes-drift velocity field is *divergent*, on account of the E_T contribution to $\bar{w}^{\mathrm{S}}$:

$$\mu\frac{\partial \bar{u}^{\mathrm{S}}}{\partial X} + \frac{\partial \bar{w}^{\mathrm{S}}}{\partial z} = \frac{2\mu c_{\mathrm{g}} E_T}{\rho c^3}\cos(2mz). \tag{A 2}$$

This was first pointed out to the writer by D. G. Andrews (personal communication). The Lagrangian-mean velocity $\bar{\mathbf{u}}^{\mathrm{L}}$ is therefore also divergent, satisfying (A 2). Explicitly, (11*a*, *b*) plus (A 1*a*, *b*) gives, using (12),

$$\bar{u}^{\mathrm{L}} = \bar{u} + \bar{u}^{\mathrm{S}} = 2(\gamma - c_{\mathrm{g}}^2/c^2)(E/\rho c)\cos(2mz), \tag{A 3a}$$

$$\bar{w}^{\mathrm{L}} = \bar{w} + \bar{w}^{\mathrm{S}} = -\mu m^{-1}\gamma(E_X/\rho c)\sin(2mz) \tag{A 3b}$$

$$[\gamma \equiv 2c_{\mathrm{g}}^3(c^2 - 2cc_{\mathrm{g}} - 2c_{\mathrm{g}}^2)/c^2(c^3 - 4c_{\mathrm{g}}^3)].$$

An independent check on (A 3*b*) and thus on (A 1*b*) is obtained by calculating direct from (5) and (11*d*) the Lagrangian-mean vertical displacement

$$\bar{\zeta}^{\mathrm{L}} = -N^{-2}(\bar{\theta} + \overline{\boldsymbol{\xi}.\nabla\theta}) = c_{\mathrm{g}}^{-1}m^{-1}\gamma(E/\rho c)\sin(2mz), \tag{A 4}$$

correct to order a^2. This yields an expression for $\bar{w}^{\mathrm{L}} = \mu\partial\bar{\zeta}^{\mathrm{L}}/\partial T = -\mu c_{\mathrm{g}}\partial\bar{\zeta}^{\mathrm{L}}/\partial X$, which agrees with (A 3*b*).

Thus it appears that in waveguide problems such as the present one, with a transverse length scale $H \not\gg k^{-1}$, the effect of the waves on the Lagrangian-mean flow is not equivalent to a stress alone. There is also an apparent 'mean mass' source, given in this problem by the right-hand side of (A 2). One could if one wished continue to define a 'radiation stress' in the way proposed by Bretherton (1971, §6.5), but this would share, with the Reynolds stress of the Eulerian-mean problem, the disadvantage of not representing the sole effect of the waves on the mean flow.

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